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# ON THE LAGRANGE AND JACOBI PRINCIPLES FOR NONHOLONOMIC SYSTEMS

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It was shown in [1—4] that the integral variational principles of Hamilton, Lagrange, and Jacobi are valid for holonomic, as well as nonholonomic systems, although in the case of the latter the comparison curves do not, generally, satisfy the nonintegrable constraint equations. Hence, generally speaking, they are not principles of stationary action for nonholonomic systems. The necessary and sufficient conditions of existence of solutions of the equations of motion of nonholonomic systems among solutions of Euler's equations of the Lagrange problem were established in [5] for the Hamilton principle, i.e. when in the first approximation it is the principle of stationary action. The similar problem for the Lagrange and the Jacobi principles is solved here for mechanical systems subjected to nonholonomic stationary constraints homogeneous with respect to generalized velocities, and acted upon by potential forces defined by derivatives of the generalized force function. Necessary and sufficient conditions for the Lagrange and Jacobi principles to be principles of stationary action are established. These conditions coincide with those in [5]. Conditions under which the theorem and the energy integral of systems subjected to ideal nonlinear nonholonomic constraints are also formulated, and conditions under which real displacements of a nonholonomic system can be found among possible displacements are indicated.

1. Let us consider a mechanical system under nonintegrable ideal constraints

$$f_l(q_i, q_i^{\cdot}, t) = 0 \quad (l = 1, \ldots, r) \tag{1.1}$$

nonlinear, in the general case, relative to the generalized velocities $q_i^{\cdot} = dq_i / dt$, where $q_i$ $(i = 1, \ldots, n)$ are the system's Lagrange coordinates and $t$ is time. Constraints (1.1) are assumed independent, i.e.,

$$\operatorname{rank}\left\|\frac{\partial f_l}{\partial q_i^{\cdot}}\right\| = r$$

One of the fundamental principles in the dynamics of mechanical systems is the D'Alembert—Lagrange principle which in generalized coordinates has the form

$$\sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} - Q_i\right)\delta q_i = 0 \tag{1.2}$$

Here $L(q_i, q_i^{\cdot}, t) = T + U$ is the Lagrange function, $T(q_i, q_i^{\cdot}, t)$ is the kinetic energy, $U = U_1(q_i, q_i^{\cdot}, t) + U_0(q_i, t)$ is the generalized force function, where the function $U_1(q_i, q_i, t)$ is a linear form in the velocities $q_i^{\cdot}$, $Q_i$ are the generalized nonpotential forces, and $\delta q_i$ are the feasible (virtual) displacements satisfying the Chetaev conditions

$$\sum_{i=1}^{n} \frac{\partial f_l}{\partial q_i^{\cdot}} \delta q_i = 0 \qquad (l = 1, \ldots, r) \tag{1.3}$$

Function $L$ is of the second degree in the generalized velocities, $L = L_2 + L_1 + L_0$, where $L_2 = T_2$ is a positive-definite quadratic form, $L_1 = T_1 + U_1$ is a linear form in the velocities $q_i^{\cdot}$, and $L_0 = T_0 + U_0$ is independent of $q_i^{\cdot}$.

If the real displacements $dq_i = q_i^{\cdot}\, dt$ of the system are found among the feasible displacements $\delta q_i$, then from relation (1.2) follows the energy theorem

$$\frac{d}{dt}\Big(\sum_{i=1}^{n} q_i^{\cdot} \frac{\partial L}{\partial q_i^{\cdot}} - L\Big) = \sum_{i=1}^{n} Q_i q_i^{\cdot} - \frac{\partial L}{\partial t} \tag{1.4}$$

of the same form as for holonomic systems when constraints (1.1) are absent. If the nonpotential forces are gyroscopic or are absent, while the Lagrange functions does not depend explicitly on time, i.e., under the conditions

$$\sum_{i=1}^{n} Q_i q_i^{\cdot} = 0, \qquad \frac{\partial L}{\partial t} = 0 \tag{1.5}$$

we obtain the generalized energy integral

$$\sum_{i=1}^{n} q_i^{\cdot} \frac{\partial L}{\partial q_i^{\cdot}} - L = L_2 - L_0 = T_2 - T_0 - U_0 = h = \text{const} \tag{1.6}$$

from equality (1.4). Integral (1.6) becomes the physical energy integral

$$T - U_0 = h \tag{1.7}$$

if the geometric (finite) constraints imposed on the system are stationary (then $T = T_2$, $T_1 = T_0 = 0$).

Incorrect assertions are encountered even in textbooks when discussing the conditions when the real displacements of a nonholonomic system are to be found among its feasible displacements. Thus (see [6] ), it is asserted that "under differential constraints, just as under finite constraints, the virtual displacements coincide with the true ones if the constraint is scleronomous, i.e., the equation of the differential constraint has the form

$$\sum_{v=1}^{n} (a_{pv}\, dx_v + b_{pv}\, dy_v + c_{pv}\, dz_v) = 0$$

where $a_{pv}$, $b_{pv}$, $c_{pv}$ are functions only of the coordinates and not of time occurring explicitly". However, it is not difficult to see [4, 7] that in the case of rheonomic homogeneous constraints of such kind, when the coefficients $a_{pv}$, $b_{pv}$, $c_{pv}$ do depend explicitly on time, the real displacements are found among the feasible ones. Regarding nonlinear constraints of form (1.1), their time independence does not ensure the possession of this property in the general case. As a matter of fact, in order that the real displacements $dq_i = q_i^{\cdot} dt$ be found among the feasible ones satisfying conditions (1.3), it is necessary and sufficient that conditions

$$\sum_{i=1}^{n} \frac{\partial f_l}{\partial q_i^{\cdot}} q_i^{\cdot} = 0 \quad (l = 1, \ldots r) \tag{1.8}$$

be fulfilled. These conditions are fulfilled for constraints homogeneous in $q_i^{\cdot}$ since by Euler's theorem on homogeneous functions we have

$$\sum_{i=1}^{n} \frac{\partial f_l}{\partial q_i^{\cdot}} q_i^{\cdot} = k_l f_l (q_i, q_i^{\cdot}, t)$$

independently of the stationarity of constraint (1.1); $k_l$ is the degree of homogeneity of function $f_l$. Conversely, conditions (1.8) are, generally speaking not fulfilled for inhomogeneous constraints even if functions (1.1) are time-independent. From what has been said it is obvious that in the general case one cannot carry over to nonholonomic systems the assertion, valid for holonomic systems, that if the constraints do not depend explicitly on time, the real displacements are to be found among the feasible ones. We remark that in the case of homogeneous constraints (1.1) the class of feasible velocities includes the rest state $q_i^{\cdot} = 0$, in connection with which systems under nonlinear homogeneous constraints can be referred to the category of catastatic systems [4].

Let us now consider the inverse problem. Assume that the equations of motion of a nonholonomic system

$$\frac{d}{dt} \frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = Q_i + \sum_{l} \mu_l \frac{\partial f_l}{\partial q_i^{\cdot}} \tag{1.9}$$

where $\mu_l$ are Lagrange multipliers, admit of energy integral (1.6). Differentiating (1.6) with respect to $t$ relative to (1.9), we find that for relation (1.6) to be, under conditions (1.5), an integral of Eqs. (1.9), it is necessary and sufficient that the identity

$$\sum_{l, i} \mu_l \frac{\partial f_l}{\partial q_i^{\cdot}} q_i^{\cdot} = 0 \tag{1.10}$$

be fulfilled under conditions (1.1). Condition (1.10) is fulfilled for homogeneous constraints, but, generally speaking, it is not fulfilled for inhomogeneous constraints. Thus, for example, if only one constraint of form (1.1) is imposed on the system, then condition (1.10) with $\mu \neq 0$ reduces to one equality of form (1. 8), not fulfilable in general case for an inhomogeneous constraint $f(q_i, q_i^{\cdot}, t) = 0$.

2. From now on we assume that only potential forces possessing either a generalized force function $U = U_1 + U_0$ or a force function $U = U_0$ act on the system, so that all nonpotential force $Q_i = 0$ $(i = 1, \ldots, n)$, the Lagrange function $L = T + U$ does not depend explicitly on time and the constraints (1.1) are homogeneous in $q_i^{\cdot}$ and do not depend explicitly on time, i.e., are

$$f_l (q_i, q_i^{\cdot}) = 0 \quad (l = 1, \ldots, r) \tag{2.1}$$

Under these conditions the generalized energy integral (1.6) or the energy integral (1.7) holds. The existence of integral (1.6) or (1.7) permits use, as is well known [1], when considering an integral variational principle of least action, to restrict the set of comparable motions leading the system from one position to another, in which the energy has one and the same fixed value $h$.

Let us consider the real motion of the system between some initial position $P_0$ and final position $P_1$, for which the constant $h$ of the generalized energy integral (1.6) has a specific value. If we compare this motion with sufficiently proximate variational motions between the same initial $P_0$ and final $P_1$ positions, taking place, with Eq. (1.6) observed, with the same value of constant $h$ of generalized energy as in the real motion, then for the latter, by the Lagrange principle [8]

$$\Delta \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i^{\cdot}} q_i^{\cdot} dt = 0 \qquad (2.2)$$

Here $\Delta$ is the symbol of complete (asynchronous) variation, and it is assumed that at the initial position $P_0$ and final position $P_1$, passed through at instants $t_0$ and $t_1$, all

$$\Delta q_i = 0 \quad (i = 1, \ldots, n) \qquad (2.3)$$

The instants at which the system passes through positions $P_0$ and $P_1$ are not fixed; they depend upon the curve along which the system moves, i.e., in the general case, $\Delta t_0 \neq 0$ and $\Delta t_1 \neq 0$. The complete and the synchronous (virtual) variations of the coordinates are connected by the relation

$$\Delta q_i = \delta q_i + q_i^{\cdot} \Delta t \qquad (2.4)$$

applicable as well to any differentiable function of the coordinates and time. We henceforth assume that the variations $\Delta q_i$ and $\Delta t$ are function of $t$ of class $C_2$.

We note that by virtue of integral (1.6) the Lagrange principle (2.2) can be written as well in the form [4]

$$\Delta \int_{t_0}^{t_1} (L + h) dt = 0 \qquad (2.5)$$

under conditions (2.3) and (1.6). The Lagrange principle in form (2.2) or (2.5) is valid both for holonomic systems as well as for nonholonomic generalized-conservative or conservative systems [1—4]. We can convince ourselves of this, for example, by deriving the Eqs. (1.9) of motion of nonholonomic systems (with $Q_i = 0$) from (2.2) or (2.5). However, for nonholonomic systems the variational motions do not, generally speaking comply with constraint equations (2.1); in view of this the writing of (2.2) or (2.5) has a conditional sense [4]: the comparison curves do not satisfy Eqs. (2.1), whereas a real trajectory defined from (2.2) or (2.5) does satisfy these equations.

In this regard the Lagrange principle, as the Hamilton principle, for nonholonomic systems is in general case, not a principle of stationary action in the sense of the calculus of variations. Under specified conditions [5], however, the Hamilton principle for nonholonomic systems can have, in the first approximation, the character of the principle of stationary action. Since the Lagrange principle is closely related to the

Hamilton principle [1], we can expect that under the conditions mentioned the Lagrange principle will also have, in the first approximation, the character of the principle of stationary action. In order to be convinced of this we consider the equations of the extremals of variational problem (2.2) in the class of curves satisfying conditions (2.1) and (1.6). This problem on conditional extremum reduces to a problem on unconditional extremum

$$\Delta \int_{t_0}^{t_1} F dt = 0 \tag{2.6}$$

under conditions (2.3). Here the integrand is

$$F = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i^{\cdot}} q_i^{\cdot} + \lambda \Big( \sum_{i=1}^{n} \frac{\partial L}{\partial q_i^{\cdot}} q_i^{\cdot} - L - h \Big) + \sum_{l=1}^{r} \varkappa_l f_l (q_i, q_i^{\cdot}) \tag{2.7}$$

where $\lambda$ and $\varkappa_l$ are undetermined multipliers, being certain time functions. It is easy to perceive [8] the validity of the equality

$$\Delta \int_{t_0}^{t_1} F dt = \Big[ \Big( F - \sum_{i=1}^{n} \frac{\partial F}{\partial q_i^{\cdot}} q_i^{\cdot} \Big) \Delta t + \sum_{i=1}^{n} \frac{\partial F}{\partial q_i^{\cdot}} \Delta q_i \Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_{i=1}^{n} \Big( \frac{d}{dt} \frac{\partial F}{\partial q_i^{\cdot}} - \frac{\partial F}{\partial q_i} \Big) \delta q_i dt$$

Here all variations $\delta q_i$ are taken to be arbitrary and independent, while the $\Delta q_i$ satisfy conditions (2.3), as a consequence of which we obtain from equality (2.6) the equations for the extremals and the transversality conditions at the endpoints of the extremals

$$\frac{d}{dt} \frac{\partial F}{\partial q_i^{\cdot}} - \frac{\partial F}{\partial q_i} = 0 \quad (i = 1, \ldots, n) \tag{2.8}$$

$$F - \sum_{i=1}^{n} \frac{\partial F}{\partial q_i^{\cdot}} q_i^{\cdot} = 0 \quad \text{for } t = t_0, t_1 \tag{2.9}$$

Since by virtue of Eqs. (2.8) the time derivative in the left-hand side of (2.9) is zero, that side of (2.9) is constant and equal zero along the whole of the extremum

$$F - \sum_{i=1}^{n} \frac{\partial F}{\partial q_i^{\cdot}} q_i^{\cdot} = 0$$

Substituting the expression of function (2.7) into this equation, with due regard to Eqs. (1.6) and (2.1) we obtain the equality

$$-(1 + \lambda) \sum_{i,j=1}^{n} \frac{\partial^2 L}{\partial q_i^{\cdot} \partial q_j^{\cdot}} q_i^{\cdot} q_j^{\cdot} = 0$$

from which we find $\lambda = -1$ since by assumption $\| \partial^2 L / \partial q_i^{\cdot} \partial q_j^{\cdot} \| \neq 0$. Consequently, function (2.7) takes the form

$$F = L + h + \sum_l \varkappa_l f_l (q_i, q_i^{\cdot})$$

Substituting this expression into Eq. (2.8), we obtain the equations for the extremals of problem (2.6)

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = \sum_l \varkappa_l \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt}\frac{\partial f_l}{\partial q_i^{\cdot}}\right) - \sum_l \varkappa_l^{\cdot} \frac{\partial f_l}{\partial q_i^{\cdot}} \quad (i = 1, \ldots, n) \qquad (2.10)$$

These equations coincide with the Eqs. (3.2) in [5] for the extremals of the variational Lagrange problem for the Hamilton principle. A comparison of Eqs. (2.10) with Eqs. (1.9) for the motion of the nonholonomic system (with $Q_i = 0$ leads to the condition

$$\sum_{l,i} \varkappa_l \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt}\frac{\partial f_l}{\partial q_i^{\cdot}}\right) \delta q_i = 0 \qquad (2.11)$$

necessary and sufficient for the solution of Eqs. (1.9) and (2.1) to be found among the solutions of Eqs. (2.10) and (2.1). Thus, for the motions of the nonholonomic system, satisfying condition (2.11), the Lagrange principle has, in the first approximation, the character of the principle of stationary action.

3. In order to bypass the difficulties connected with asynchronous variation we can, following Jacobi [9], select as the independent variable a certain parameter $\lambda$ constinuously and monotonely varying between constant values $\lambda_0$ and $\lambda_1$ corresponding to the system's positions $P_0$ and $P_1$. In the system's motion the variables $q_i$, $q_i^{\cdot}$ and $t$ are functions of this parameter $\lambda$. We denote the derivative of $q_i$ with respect to $\lambda$ by $q_i'$, so that

$$q_i^{\cdot} = q_i' d\lambda / dt$$

The constraint Eqs. (2.1), homogeneous in $q_i^{\cdot}$, become

$$f_l(q_i, q_i') = 0 \quad (l = 1, \ldots, r)$$

and the feasible displacements $\delta q_i$ (for a fixed $\lambda$ ) must, by virtue of (1.3), satisfy the conditions

$$\sum_{i=1}^{n} \frac{\partial f_l}{\partial q_i'} \delta q_i = 0 \quad (l = 1, \ldots, r) \qquad (3.1)$$

If the system's real motion between certain initial $P_0$ and final $P_1$ positions, for which the constant $h$ of generalized energy integral (1.6) has a specific value, is compared with sufficiently proximate variational motions between the same positions $P_0$ and $P_1$, taking place with the same generalized energy $h$, as in the real motion, then for the latter, by the Jacobi principle

$$\delta \int_{\lambda_0}^{\lambda_1} \left(\sqrt{2(h + L_0)}\sqrt{2\theta} + \Phi\right) d\lambda = 0 \qquad (3.2)$$

and

$$\delta q_i = 0 \quad \text{for} \quad \lambda = \lambda_0, \lambda_1 \qquad (3.3)$$

The functions $\theta(q_i, q_i')$ and $\Phi(q_i, q_i')$ are defined by the formulas

$$\theta = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q_s) q_i' q_j', \quad \Phi = \sum_{i=1}^{n} a_i(q_s) q_i'$$

if the quadratic form $L_2$ and linear form $L_1$ entering into the Lagrange function $L$ are specified as

$$L_2 = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q_s)\, q_i^{\cdot} q_j^{\cdot}, \quad L_1 = \sum_{i=1}^{n} a_i(q_s)\, q_i^{\cdot}$$

Obviously, the relations

$$L_2 = \theta\left(\frac{d\lambda}{dt}\right)^2, \quad L_1 = \Phi\frac{d\lambda}{dt}$$

are valid, and from integral (1. 6) follows

$$\frac{d\lambda}{dt} = \sqrt{\frac{h+L_0}{\theta}} \tag{3.4}$$

With due regard to (3. 1), from the Jacobi principle (3. 2) we can obtain the differential equations for the system's real path

$$\frac{d}{d\lambda}\left(\frac{\sqrt{2(h+L_0)}}{\sqrt{2\theta}}\frac{\partial\theta}{\partial q_i'} + \frac{\partial\Phi}{\partial q_i'}\right) - \frac{\sqrt{2\theta}}{\sqrt{2(h+L_0)}}\frac{\partial L_0}{\partial q_i} - \tag{3.5}$$

$$\frac{\partial\Phi}{\partial q_i} - \frac{\sqrt{2(h+L_0)}}{\sqrt{2\theta}}\frac{\partial\theta}{\partial q_i} = \sum_{l=1}^{r}\mu_l\frac{\partial f_l}{\partial q_i'} \quad (i = 1, \ldots, n)$$

We see that by the replacement of variable $\lambda$ by $t$ in accordance with (3. 4), we can, with due regard to (1. 6) take Eqs. (3. 5) into the for of the equations of motion (1. 9) with $Q_i = 0 \quad (i = 1, \ldots, n)$.

We now consider the equations for the extremals of variational problem (3. 2) in the class of curves satisfying constraint Eqs. (2. 1). This problem on conditional extremum reduces to the problem on unconditional extremum

$$\delta\int_{\lambda_0}^{\lambda_1}\Psi d\lambda = 0 \tag{3.6}$$

under conditions (3. 3). Here the integrand is

$$\Psi = \sqrt{2(h+L_0)}\sqrt{2\theta} + \Phi + \sum_l \varkappa_l f_l(q_i, q_i')$$

The Euler equations for problem (3. 6) are

$$\frac{d}{d\lambda}\left(\frac{\sqrt{2(h+L_0)}}{\sqrt{2\theta}}\frac{\partial\theta}{\partial q_i'} + \frac{\partial\Phi}{\partial q_i'}\right) - \frac{\sqrt{2\theta}}{\sqrt{2(h+L_0)}}\frac{\partial L_0}{\partial q_i} - \frac{\partial\Phi}{\partial q_i} - \tag{3.7}$$

$$\frac{\sqrt{2(h+L_0)}}{\sqrt{2\theta}}\frac{\partial\theta}{\partial q_i} = \sum_l \varkappa_l\left(\frac{\partial f_l}{\partial q_i} - \frac{d}{d\lambda}\frac{\partial f_l}{\partial q_i'}\right) - \sum_l \varkappa_l'\frac{\partial f_l}{\partial q_i'}$$

$$(i = 1, \ldots, n)$$

Comparing Eqs. (3. 5) and (3. 7) are repeating the arguments (see [5] ), we conclude that the condition

$$\sum_{l,i}\varkappa_l\left(\frac{\partial f_l}{\partial q_i} - \frac{d}{d\lambda}\frac{\partial f_l}{\partial q_i'}\right)\delta q_i = 0 \tag{3.8}$$

is necessary and sufficient for finding some solution of Eqs. (3.5) and (2.1) among the solutions of Eqs. (3.7) and (2.1). Condition (3.8) is obviously equivalent to condition (2.11).

Thus, for the motions of the nonholonomic system, satisfying condition (2.11), the Jacobi principle, has, in the first approximation, the character of the principle of stationary action.

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